1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that for every $x \in \mathbb{R}, f^{-1}(x)$ is countable. Prove that there are uncountably many $\alpha \in \mathbb{R}$ such that $f(\alpha)$ is irrational.

   Solution: On the contrary assume that the set $\{\alpha \in \mathbb{R} : f(\alpha) \in \mathbb{Q}^c\}$ is countable. Since countable union of countable sets is countable. Therefore, the set $\bigcup_{x\in\mathbb{Q}} f^{-1}(x)$ is countable. Since,

$$\bigcup_{x\in\mathbb{Q}} (f^{-1}(x))^c = \bigcap_{x\in\mathbb{Q}} (f^{-1}(x))^c = \bigcap_{x\in\mathbb{Q}} \{y \in \mathbb{R} | f(y) \neq x\} = \{y \in \mathbb{R} : f(y) \in \mathbb{Q}^c\}$$

   By our assumption, the set $\bigcup_{x\in\mathbb{Q}}(f^{-1}(x))^c$ is countable. This is a contradiction to the fact that $\mathbb{R}$ is uncountable. Hence, there are uncountably many $\alpha \in \mathbb{R}$ such that $f(\alpha)$ is irrational.

2. Prove or disprove the following statement: Any total order on $\mathbb{N}$ is a well order.
   This means you should either provide a proof of the above statement or disprove it by giving an example of a total order on $\mathbb{N}$ which is not a well order.

   Solution: Every total order on $\mathbb{N}$ is not necessarily a well order. In the following we give a relation on $\mathbb{N}$, which is totally order but not well order. For $a, b \in \mathbb{N}$, define $a \leq b$ iff $a \geq b$. Here $(\mathbb{N}, \leq)$ is a total ordered set but not well ordered.

3. verify that 293 is invertible in $\mathbb{Z}_{929}$ and find its multiplicative inverse.

   Solution: Clearly, $\mathbb{Z}_{929}$ is cyclic. Hence, the element 293 is invertible. Let x be the inverse of 293 in $\mathbb{Z}_{929}$. Therefore,

$$\begin{aligned} 293x &\equiv 1 \mod 929 \\ 293x &= 929k + 1 \\ 293x &= 879k + 50k + 1 \end{aligned}$$

   We have that $50k + 1$ is divisible by 293 for some $k \in \mathbb{N}$. Take $k = 41$. We have

$$\begin{aligned} 293x &= 293(3k) + 293(7) \\ 293x &= (3k + 7)293 \end{aligned}$$

   Hence $x = 3(41) + 7 = 130$

4. Let a_n denote the number of elements of the set $\{\sigma \in S_7 | o(\sigma) = n\}$. where S_7 is the symmetric group on 7 symbols. Calculate a_2, a_6, a_8 and a_{12}.

   Solution: We need to find the number of elements of order 2 in S_7. The order of a permutation expressed as a product of disjoint cycles is the lcm of length of cycles.
   Elements of order 2:
   We have the following possobilities for elements of order 2, when expressed as a disjoint cycles.

$(a)(bc)(de)(fg), (ab)(c)(d)(e)(f)(g)$ and $(ab)(cd)(e)(f)(g)$.
For the first type, there are 7 ways to pick a, 6 ways to pick b, 4 ways to pick c and so on. Since (bc) and (cb) represent the same permutation and there are three 2-cycles so we need to divide by 2^3. The permutation $(bc)(de)(fg)$ can be written in 3! different ways. we need to divide by 3! to avoid any repetition. Therefore total number of elements of the cycle type $(a)(bc)(de)(fg)$ are $\frac{7!}{2^3 3!} = 105$.
Similarly, total number of elements of cycle type $(ab)(c)(d)(e)(f)(g)$ are $\frac{7!}{(2)(5!)} = 21$.
Similarly, total number of elements of cycle type$(ab)(cd)(e)(f)(g)$ are $\frac{7!}{2^2(2!)(3!)} = 105$.

Therefore, total number of elements of order 2 in S_7 are: $105 + 105 + 21 = 231$.

Elements of order 6:
Possible cycle types of elements of order 6 are : $(abc)(de)(fg)$, $(abc)(de)(f)(g)$, $(abcdef)(g)$.
For the first type, there are $\binom{7}{3}$ ways to pick a, b and c. There are 3! ways to arrange these elements. Since (abc), (bca) and (cab) represent the same permutation. We need to divide by 3 to avoid any repetition. There are $\binom{4}{2}$ to pick d and e and there is only one way to chose f and g. So the total number of elemnets of this cysle type are $\binom{7}{3}\frac{3!}{3}\binom{4}{2}\frac{1}{2} = 210$.

For the second type: There are $\binom{7}{3}$ ways to pick a, b and c and as above there are we will multiply by 3! to arrange them and divide by 3 to avoid repetition. There are $\binom{4}{2}$ ways to pick d and e, and as above we will divide by 2. and there are 2 ways to pick f and 1 way to pick g. So the total number of elements of this cycle type are $\binom{7}{3}\frac{3!}{3}\binom{4}{2}\frac{2}{2} = 420$.

For the third type, there are 7 ways to chose g and 6! ways to arrange remaining elements in one cycle. In this way, 6 Six-cycles will represent the same permutation. So we will divide by 6. Therefore, the total number of elements of this cycle type are $7\frac{6!}{6} = 840$.

Hence, the total number of elements of order 6 in S_7 are: $210 + 420 + 840 = 1470$.

Elements of order 8: Clearly, there is no elements of order 8 in S_7. Since, there is no way to make an 8-cycle from elements coming from S_7.

Elements of order 12: Possible cycle types of elements of order 12 are: $(abc)(defg)$. There are $\binom{7}{3}\frac{3!}{3}$ ways to pick three elements and arrange them to get 3-cycles. Similarly, there are $\binom{4}{4}\frac{4!}{4}$ ways to make 4-cycles. Hence the total number of elements of this cycle type are $\binom{7}{3}\frac{3!}{3}\binom{4}{4}\frac{4!}{4} = 420$.

5. (a) If every element of a group is of order 2. Prove that G is abelian.
Solution Every element of the group G is self- inverse. For every $a, b \in G$, consider,

$$\begin{aligned} (ab)^{-1} &= ab \\ b^{-1}a^{-1} &= ab \\ ba &= ab \end{aligned}$$

Therefore, G is abelian.

(b) Give an example of a group G and two elements $a, b \in G$ such that $o(a) = o(b) = 2$ while $o(ab) = 6$.
Solution Take $G = S_6$. Let $a = (12)$ and $b = (56)(42)$. Then $ab = (124)(56)$. Clearly, $o(a) = o(b) = 2$ and $o(ab) = 6$.

6. Let G be a group of order n. Prove that every subgroup of G is cyclic and for every positive integer d dividing n, there is a unique subgroup of order d.

   Solution Please refer to Theorem 4.3 of Contemporary Abstract Algebra by Joseph A. Gallian.